Ramsey Numbers for the Path with Three Edges

JÜRGEN BIERBRAUER

Let P_3 denote the path with 3 edges. The *Ramsey number* $r(P_3, k)$, k a natural number, is the minimum number of vertices in a complete graph for which every k-colouring of lines will produce a monochromatic copy of P_3. We fill the last gap in the determination of these numbers by proving the following.

THEOREM 1. $r(P_3, 3^m) = 2\cdot 3^m + 1$, *if* $m > 1$

In [1] we introduced another generalization of the concept of Ramsey number: $r(\mathcal{F}_n, k)$ is defined as the minimum number of vertices of a complete graph such that every k-colouring of the edges produces a monochromatic connected component on more than n vertices. Obviously $r(\mathcal{F}_3, k) \leq r(P_3, k)$ for every k. The construction used to prove Theorem 1 also yields the following

THEOREM 2. $r(\mathcal{F}_3, k) = 2k + 1$, *if* $k = 3^m h$, $m \geq 1$, $h \equiv 1 \pmod 3$, $k > 3$.

From the works of Irving and Lindström [3, 2] we know the numbers $r(P_3, k)$ if $k \neq 3^m$. Further $r(P_3, 3^m) \in \{2\cdot 3^m, 2\cdot 3^m + 1\}$. As $r(P_3, 3) = 6$, it was natural to conjecture that $r(P_3, 3^m) = 2\cdot 3^m$ for all m ([3]). However, the contrary turns out to be true.

Consider a putative 9-colouring of edges of the complete graph K_{18}, which avoids monochromatic copies of P_3. Easy counting arguments show, that this must arise out of a *Steiner triple system* (in abbreviated STS) γ on 19 points by deleting one point P. Thus we have to look for an STS with a point P, such that the 'punctured' STS $\gamma - \{P\}$ is *resolvable* as a linear space, i.e. the blocks of γ-$\{P\}$ are partitioned in 9 classes, each class yielding a partition of the points. The reverse STS are natural candidates. An STS is called *reverse* if it possesses an involutory automorphism with exactly one fixed point. The reverse STS on 19 points given in [4] does the job. Its points are P, 1, 2, ..., 9, $\bar{1}$, $\bar{2}$, ..., $\bar{9}$. The permutation which fixes P and interchanges the points i and $\bar{i}$, $i = 1, 2, \ldots, 9$ is an automorphism of γ. The triples fall into 9 classes C, $T_1, \ldots, T_8$:

C: $(P, i, \bar{i})$, $i = 1, 2, \ldots, 9$.
T_1: $(1,4,7), (2,5,8), (3,6,9), (\bar{1},\bar{4},\bar{7}), (\bar{2},\bar{5},\bar{8}), (\bar{3},\bar{6},\bar{9})$
T_2: $(1,5,9), (2,6,7), (3,4,8), (\bar{1},\bar{5},\bar{9}), (\bar{2},\bar{6},\bar{7}), (\bar{3},\bar{4},\bar{8})$
T_3: $(1,2,\bar{6}), (\bar{1},\bar{2},6), (4,5,\bar{9}), (\bar{4},\bar{5},9), (3,\bar{7},\bar{8}), (\bar{3},7,8)$
T_4: $(1,3,\bar{5}), (\bar{1},\bar{3},5), (4,6,\bar{8}), (\bar{4},\bar{6},8), (2,\bar{7},\bar{9}), (\bar{2},7,9)$
T_5: $(1,\bar{8},\bar{9}), (\bar{1},8,9), (2,3,\bar{4}), (\bar{2},\bar{3},4), (5,6,\bar{7}), (\bar{5},\bar{6},7)$
T_6: $(1,\bar{3},6), (\bar{1},3,\bar{6}), (2,\bar{8},9), (\bar{2},8,\bar{9}), (4,\bar{5},\bar{7}), (\bar{4},5,7)$
T_7: $(1,\bar{2},\bar{4}), (\bar{1},2,4), (3,7,\bar{9}), (\bar{3},\bar{7},9), (5,\bar{6},\bar{8}), (\bar{5},6,8)$
T_8: $(1,\bar{7},8), (\bar{1},7,\bar{8}), (2,\bar{3},\bar{5}), (\bar{2},3,5), (4,\bar{6},9), (\bar{4},6,\bar{9})$

This shows $r(P_3, 9) = r(\mathcal{F}_3, 9) = 19$ (see the upper bound $r(\mathcal{F}_n, k) \leq k(n-1)+1$ for $k \equiv 0 \pmod n$ given in [1]).

Lindström's inductive process as introduced in [3] now yields our Theorems. This process uses the existence of pairs of orthogonal latin squares for every order $k \notin \{1, 2, 6\}$.

A beautiful short proof of this fact has recently been given by Zhu Lie [5].

We collect all the numbers in question together into a final Theorem.

0195-6698/86/030205+02 $02.00/0

THEOREM.

$$(a)\qquad r(P_3, k) = \begin{cases} 2k+2, & \text{if } k \equiv 1 (\text{mod } 3), \\ 2k+1, & \text{if } k \equiv 0 \text{ or } 2 (\text{mod } 3),\ k \neq 3, \\ 6, & \text{if } k = 3, \end{cases}$$

$$(b)\qquad r(\mathscr{F}_3, k) = \begin{cases} 2k+2, & \text{if } k \equiv 1 (\text{mod } 3), \\ 2k, & \text{if } k = 3 \text{ or } k \equiv 2 (\text{mod } 3), \\ 2k+1, & \text{if } k = 3^m h > 3,\ h \equiv 1 (\text{mod } 3). \end{cases}$$

REFERENCES

1. J. Bierbrauer, A. Brandis, On generalized Ramsey numbers for trees, *Combinatorica* **5** (1985), 95–107.
2. R. Irving, Generalized Ramsey numbers for small graphs, *Discr. Math.* **9** (1974), 251–264.
3. B. Lindström, Undecided Ramsey numbers for paths, *Discr. Math.* **43** (1983), 111–112.
4. A. Rosa, On reverse Steiner triple systems, *Discr. Math.* (1) **2** (1972), 61–71.
5. Zhu Lie, 'A short disproof of Euler's conjecture concerning orthogonal Latin squares', *Ars Comb.* **14** (1982), 47–55.

Received 25 January 1985
J. BIERBRAUER
Mathematisches Institut der Universität,
Im Neuenheimer Feld 288, 69 Heidelberg, B.R.D.